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2000 J. Phys. A: Math. Gen. 33 2545

(http://iopscience.iop.org/0305-4470/33/13/307)

J. Phys. A: Math. Gen. **33** (2000) 2545–2553. Printed in the UK PII: S0305-4470(00)10471-8

Fock representations of the superalgebra $sl(n+1|m)$, its quantum analogue $U_q[sl(n+1|m)]$ and related quantum statistics

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Received 20 December 1999

Abstract. Fock space representations of the Lie superalgebra $sl(n+1|m)$ and of its quantum analogue $U_q[sl(n+1|m)]$ are written down. The results are based on a description of these superalgebras via creation and annihilation operators. The properties of the underlying statistics are briefly discussed.

1. Introduction

The quantization of the simple Lie algebras [1] and the basic Lie superalgebras [2] is usually carried out in terms of their Chevalley generators $e_i, f_i, h_i, i = 1, \ldots, n$ for an algebra of rank n. Recently, it was realized that the quantum algebras $U_q[osp(1|2n)]$ [3], $U_q[so(2n+1)]$ [4], $U_q[osp(2r+1|2m)]$, $r+m=n$ [5], and also $U_q[sl(n+1)]$ [6] and $U_q[sl(n+1|m)]$ [7] admit a description in terms of an alternative set of generators $a_i^\pm$, H_i, $i = 1, \ldots, n$, referred to as (deformed) creation and annihilation operators (CAOs) or generators. This certainly also holds for the corresponding non-deformed Lie superalgebras.

The concept of creation and annihilation operators of a simple Lie (super)algebra was introduced in [8]. Let $\mathcal{A}$ be such an algebra with a supercommutator $[\![\, , \,]\!]$. The root vectors $a_1^\xi, \ldots, a_n^\xi$ of $\mathcal{A}$ are said to be creation ($\xi = +$) and annihilation ($\xi = -$) operators of $\mathcal{A}$, if

$$\mathcal{A} = \mathrm{lin\,env}\{a_i^\xi,\ [\![a_j^\eta, a_k^\varepsilon]\!] \mid i, j, k = 1, \ldots, n;\ \xi, \eta, \varepsilon = \pm\} \tag{1}$$

so that $a_1^+, \ldots, a_n^+$ (respectively, $a_1^-, \ldots, a_n^-$) are negative (respectively, positive) root vectors of $\mathcal{A}$.

The Fock representations of $\mathcal{A}$, defined in [8], are constructed in a very similar way to those of Bose or Fermi operators (or their generalizations, the parabosons and parafermions

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[9]). In a more mathematical terminology the Fock modules of $\mathcal{A}$ are induced from trivial one-dimensional modules of a subalgebra $\mathcal{B}$ of $\mathcal{A}$, namely

$$\mathcal{B} = \text{lin env}\{a_i^-, [\![a_j^\eta, a_k^\varepsilon]\!] \mid i, j, k = 1, \ldots, n; \eta, \varepsilon = \pm\}. \tag{2}$$

The reason for introducing such (more physical) terminology is based on the observation that the creation and the annihilation operators of the ortho-symplectic Lie superalgebra (LS) $osp(2r+1|2m)$ have a direct physical significance: $a_1^\pm, \ldots, a_m^\pm$ (respectively $a_{m+1}^\pm, \ldots, a_n^\pm$) are para-Bose (respectively, para-Fermi) operators [10], operators which generalize the statistics of tensor (respectively, spinor) fields in quantum field theory (for $n = m = \infty$) [9]. Since $osp(2r + 1|2m)$ is an algebra from the class B in the classification of Kac [11] one may call the paraquantization a B-quantization.

It was argued in [12] that to each class A, B, C and D of basic LSs there corresponds a quantum statistics, so that their CAOs can be interpreted as creation and annihilation operators of (quasi)particles, excitations, in the corresponding Fock space(s). This assumption holds for the classes A, B, C, D of simple Lie algebras [13]. It was studied in detail for the Lie algebras $sl(n+1)$ (A-statistics) [14], for the LSs $sl(1|m)$ (A-superstatistics) [8, 15] and recently for the classical Lie superalgebra $q(n + 1)$ [16].

Here we report briefly on the properties of quantum statistics, related to the Lie superalgebra $sl(n + 1|m)$ and its quantum analogue. These statistics are particular kinds of generalized quantum statistics. There are many publications on the subject especially on the part related to quantum groups (initiated in [17]). Other interesting approaches to generalized statistics and their Fock space representations are also available. We mention here the generalizations associated with the spectral quantum Yang–Baxter equations [18], the statistics based on Lie supertriple systems [19] and on triple operator algebras [20], or extended Haldane statistics [21]. In certain approaches, one starts with a deformation of the Fock space, and from here one deduces the properties of creation and annihilation operators and the related statistics (see, e.g., [22], or [23] for ortho-Bose and ortho-Fermi statistics).

For self-consistency of the exposition we recall in section 2 the description of $sl(n+1|m)$ and $U_q[sl(n + 1|m)]$ via (deformed) creation and annihilation operators as given in [7]. The transformations of the Fock modules under the action of the CAOs are written down in section 3. In the final section we discuss briefly the underlying statistics of the creation and annihilation operators.

Throughout the paper we use the following notation. LS, LSs, Lie superalgebra, Lie superalgebras; CAOs, creation and annihilation operators; lin env, linear envelope; $\mathbb{Z}$, all integers; $\mathbb{Z}_+$, all non-negative integers; $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, the ring of all integers modulo 2; $\mathbb{C}$, all complex numbers;

$$[p; q] = \{p, p+1, p+2, \ldots, q-1, q\} \qquad \text{for} \quad p \leqslant q \in \mathbb{Z} \tag{3}$$

$$\theta_i = \begin{cases} \bar{0} & \text{if} \quad i = 0, 1, 2, \ldots, n \\ \bar{1} & \text{if} \quad i = n+1, n+2, \ldots, n+m \end{cases} \qquad \theta_{ij} = \theta_i + \theta_j \tag{4}$$

$$[a, b] = ab - ba \qquad \{a, b\} = ab + ba \qquad [\![a, b]\!] = ab - (-1)^{\deg(a)\deg(b)} ba \tag{5}$$

$$[a, b]_x = ab - xba \qquad \{a, b\}_x = ab + xba \qquad [\![a, b]\!]_x = ab - (-1)^{\deg(a)\deg(b)} xba. \tag{6}$$

2. The Lie superalgebra $sl(n + 1|m)$ and its quantization $U_q[sl(n + 1|m)]$

The universal enveloping algebra $U[gl(n + 1|m)]$ of the general linear LS $gl(n + 1|m)$ is a $\mathbb{Z}_2$-graded associative unital algebra generated by $(n + m + 1)^2$ $\mathbb{Z}_2$-graded indeterminates

$\{e_{ij}|i, j \in [0; n+m]\}$ $\deg(e_{ij}) = \theta_{ij} \equiv \theta_i + \theta_j$, subject to the relations

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk} e_{il} - (-1)^{\theta_{ij}\theta_{kl}} \delta_{il} e_{kj} \qquad i, j, k, l = 1, \ldots, n+m. \tag{7}$$

The LS $gl(n+1|m)$ is a subalgebra of $U[gl(n+1|m)]$, considered as a Lie superalgebra, with generators $\{e_{ij}|i, j \in [0; n+m]\}$ and supercommutation relations (7). The LS $sl(n+1|m)$ is a subalgebra of $gl(n+1|m)$:

$$sl(n+1|m) = \text{lin env}\{e_{ij}, (-1)^{\theta_k} e_{kk} - (-1)^{\theta_l} e_{ll} | i \neq j;\ i, j, k, l \in [0; n+m]\}. \tag{8}$$

The generators $e_{00}, e_{11}, \ldots, e_{n+m,n+m}$ constitute a basis in the Cartan subalgebra of $gl(n+1|m)$. Denote by $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n+m}$ the dual basis, $\varepsilon_i(e_{jj}) = \delta_{ij}$. The root vectors of both $gl(n+1|m)$ and $sl(n+1|m)$ are $e_{ij},\ i \neq j,\ i, j \in [0; n+m]$. The root corresponding to e_{ij} is $\varepsilon_i - \varepsilon_j$. With respect to the natural order of the basis in the Cartan subalgebra e_{ij} is a positive (respectively, a negative) root vector if $i < j$ (respectively, $i > j$).

This description of $sl(n+1|m)$ is simple, but it is not appropriate for quantum deformations. Another definition is given in terms of the Chevalley generators

$$\hat{h}_i = e_{i-1,i-1} - (-1)^{\theta_{i-1,i}} e_{ii} \qquad \hat{e}_i = e_{i-1,i} \qquad \hat{f}_i = e_{i,i-1},\ i \in [1; n+m] \tag{9}$$

and the $(n+m) \times (n+m)$ Cartan matrix $\{\alpha_{ij}\}$ with entries

$$\alpha_{ij} = (1 + (-1)^{\theta_{i-1,i}})\delta_{ij} - (-1)^{\theta_{i-1,i}} \delta_{i,j-1} - \delta_{i-1,j} \qquad i, j \in [1; n+m]. \tag{10}$$

$U[sl(n+1|m)]$ is an associative unital algebra of the Chevalley generators, subject to the Cartan–Kac and the Serre relations:

$$\begin{aligned} &[\hat{h}_i, \hat{h}_j] = 0 \qquad [\hat{h}_i, \hat{e}_j] = \alpha_{ij} \hat{e}_j \\ &[\hat{h}_i, \hat{f}_j] = -\alpha_{ij} \hat{f}_j \qquad [\![\hat{e}_i, \hat{f}_j]\!] = \delta_{ij} \hat{h}_i \end{aligned} \tag{11}$$

$$[\hat{e}_i, \hat{e}_j] = 0 \qquad [\hat{f}_i, \hat{f}_j] = 0 \qquad \text{if} \quad |i - j| \neq 1 \tag{12a}$$

$$\hat{e}_{n+1}^2 = 0 \qquad \hat{f}_{n+1}^2 = 0 \tag{12b}$$

$$[\hat{e}_i, [\hat{e}_i, \hat{e}_{i+1}]] = 0 \qquad [\hat{f}_i, [\hat{f}_i, \hat{f}_{i+1}]] = 0 \qquad i \neq n+1 \tag{12c}$$

$$[\hat{e}_{i+1}, [\hat{e}_{i+1}, \hat{e}_i]] = 0 \qquad [\hat{f}_{i+1}, [\hat{f}_{i+1}, \hat{f}_i]] = 0 \qquad i \neq n \tag{12d}$$

$$\{[\hat{e}_{n+1}, \hat{e}_n], [\hat{e}_{n+1}, \hat{e}_{n+2}]\} = 0 \qquad \{[\hat{f}_{n+1}, \hat{f}_n], [\hat{f}_{n+1}, \hat{f}_{n+2}]\} = 0. \tag{12e}$$

The grading on $U[sl(n+1|m)]$ is induced from the requirement that the only odd generators are $\hat{e}_{n+1}$ and $\hat{f}_{n+1}$. The LS $sl(n+1|m)$ is a subalgebra of $U[sl(n+1|m)]$, generated by the Chevalley generators in the sense of a Lie superalgebra.

Consider the following root vectors from $sl(n+1|m)$:

$$\hat{a}_i^+ = e_{i0} \qquad \hat{a}_i^- = e_{0i} \qquad i \in [1; n+m] \tag{13a}$$

or, equivalently

$$\begin{aligned} &\hat{a}_1^- = \hat{e}_1 \qquad \hat{a}_i^- = [[[\ldots[[\hat{e}_1, \hat{e}_2], \hat{e}_3], \ldots], \hat{e}_{i-1}], \hat{e}_i] = [\hat{a}_{i-1}^-, e_i] \qquad i \in [2; n+m] \\ &\hat{a}_1^+ = \hat{f}_1 \qquad \hat{a}_i^+ = [\hat{f}_i, [\hat{f}_{i-1}, [\ldots, [\hat{f}_3, [\hat{f}_2, \hat{f}_1]]\ldots]]] = [f_i, \hat{a}_{i-1}^+] \qquad i \in [2; n+m]. \end{aligned} \tag{13b}$$

The root of $\hat{a}_i^-$ (respectively of $\hat{a}_i^+$) is $\varepsilon_0 - \varepsilon_i$ (respectively, $\varepsilon_i - \varepsilon_0$). Therefore (with respect to the natural order of the basis $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n+m}$), $\hat{a}_1^-, \ldots, \hat{a}_{n+m}^-$ are positive root vectors, and $\hat{a}_1^+, \ldots, \hat{a}_{n+m}^+$ are negative root vectors. Moreover,

$$sl(n+1|m) = \text{lin env}\{\hat{a}_i^\xi,\ [\![\hat{a}_j^\eta, \hat{a}_k^\varepsilon]\!]\ |i, j, k \in [1; n];\ \xi, \eta, \varepsilon = \pm\}. \tag{14}$$

Hence, the generators (13) are creation and annihilation operators of $sl(n+1|m)$. These generators satisfy the following triple relations:

$$\begin{aligned}
&[\![\hat{a}_i^{\xi}, \hat{a}_j^{\xi}]\!] = 0 \qquad \xi = \pm \quad i, j \in [1; n+m] \\
&[\![[\![\hat{a}_i^{+}, \hat{a}_j^{-}]\!], \hat{a}_k^{-}]\!] = -(-1)^{\theta_{ij}\theta_k}\delta_{ik}\hat{a}_j^{-} - (-1)^{\theta_i}\delta_{ij}\hat{a}_k^{-} \\
&[\![[\![\hat{a}_i^{+}, \hat{a}_j^{-}]\!], \hat{a}_k^{+}]\!] = \delta_{jk}\hat{a}_i^{+} + (-1)^{\theta_i}\delta_{ij}\hat{a}_k^{+} \qquad i, j, k \in [1; n+m].
\end{aligned} \tag{15}$$

The CAOs (13) together with (15) define $sl(n+1|m)$ completely. The outlined description via CAOs is somewhat similar to the Lie triple system description of Lie algebras, initiated by Jacobson [24] and generalized to Lie superalgebras by Okubo [19] (see also [20] for further developments).

The relations (15) are simple. They are, however, not convenient for a quantization. It turns out that one can take only a part of these relations, so that they still completely define $sl(n+1|m)$ and are appropriate for Hopf algebra deformations.

Proposition 1 (see [7]). *$U[sl(n+1|m)]$ is an associative unital superalgebra with generators $\hat{a}_i^{\pm}$, $i \in [1; n+m]$ and relations*

$$\begin{aligned}
&[\![\hat{a}_1^{\xi}, \hat{a}_2^{\xi}]\!] = 0 \qquad [\![a_1^{\xi}, a_1^{\xi}]\!] = 0 \qquad \xi = \pm \\
&[\![[\![\hat{a}_i^{+}, \hat{a}_j^{-}]\!], \hat{a}_k^{+}]\!] = \delta_{jk}\hat{a}_i^{+} + (-1)^{\theta_i}\delta_{ij}\hat{a}_k^{+} \\
&[\![[\![\hat{a}_i^{+}, \hat{a}_j^{-}]\!], \hat{a}_k^{-}]\!] = -(-1)^{\theta_{ij}\theta_k}\delta_{ik}\hat{a}_j^{-} - (-1)^{\theta_i}\delta_{ij}\hat{a}_k^{-} \\
&|i-j| \leqslant 1 \qquad i, j, k \in [1; n+m].
\end{aligned} \tag{16}$$

The $\mathbb{Z}_2$-grading in $U[sl(n+1|m)]$ is induced from

$$\deg(\hat{a}_i^{\pm}) = \theta_i. \tag{17}$$

Passing to the quantum case, we skip the description of $U_q[sl(n+1|m)]$ via Chevalley generators. We write down directly the analogue of the relations (16). To this end we first introduce the following Cartan generators:

$$H_i = h_1 + (-1)^{\theta_1}h_2 + (-1)^{\theta_2}h_3 + \cdots + (-1)^{\theta_{i-1}}h_i. \tag{18}$$

Theorem 1 (see [7]). *$U_q[sl(n+1|m)]$ is a unital associative algebra with generators $\{H_i,\ a_i^{\pm}\}_{i\in[1;n+m]}$ and relations*

$$[H_i, H_j] = 0 \tag{19a}$$

$$[H_i, a_j^{\pm}] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^{\pm} \tag{19b}$$

$$[\![a_i^{-}, a_i^{+}]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}} \qquad L_i = q^{H_i} \quad \bar{L}_i \equiv L_i^{-1} = q^{-H_i} \quad \bar{q} \equiv q^{-1} \tag{19c}$$

$$[\![[\![a_i^{\eta}, a_{i+\xi}^{-\eta}]\!], a_k^{\eta}]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} = \eta^{\theta_k}\delta_{k,i+\xi}L_k^{-\xi\eta}a_i^{\eta} \tag{19d}$$

$$[\![a_1^{\xi}, a_2^{\xi}]\!]_q = 0 \qquad [\![a_1^{\xi}, a_1^{\xi}]\!] = 0 \qquad \xi, \eta = \pm \text{ or } \pm 1. \tag{19e}$$

3. Fock representations

We proceed to describe the Fock representations of $sl(n+1|m)$ and $U_q[sl(n+1|m)]$. The derivation, which is lengthy and non-trivial (especially in the quantum case), will be given elsewhere.

The irreducible Fock representations are labelled by one non-negative integer $p = 1, 2, \ldots,$ called an order of the statistics. To construct them assume that the corresponding representation space W_p contains (up to a multiple) a cyclic vector $|0\rangle$, such that

$$\begin{aligned} &a_i^-|0\rangle = 0 && i = 1, 2, \ldots, n+m \\ &[\![a_i^-, a_j^+]\!]|0\rangle = 0 && i \neq j = 1, 2, \ldots, n+m \\ &H_i|0\rangle = p|0\rangle && i = 1, 2, \ldots, n+m. \end{aligned} \tag{20}$$

Note that the above relations determine one-dimensional representations (labelled by p) of a subalgebra, which in the non-deformed case reduces to $\mathcal{B}$, equation (2).

From (19) one derives that the deformed creation (respectively, annihilation) operators q-supercommute

$$[\![a_i^\xi, a_j^\xi]\!]_q = 0 \qquad i < j \in [1; n+m] \qquad \xi = \pm. \tag{21}$$

This makes the basis evident (or at least one possible basis) in a given Fock space, since any product of only creation operators can always be ordered.

As a basis in the Fock space W_p take the vectors

$$\begin{aligned} |p; r_1, r_2, \ldots, r_{n+m}) &= \sqrt{\frac{[p - \sum_{l=1}^{n+m} r_l]!}{[p]![r_1]!\ldots[r_{n+m}]!}}(a_1^+)^{r_1}(a_2^+)^{r_2}\ldots(a_n^+)^{r_n} \\ &\quad \times (a_{n+1}^+)^{r_{n+1}}(a_{n+2}^+)^{r_{n+2}}\ldots(a_{n+m}^+)^{r_{n+m}}|0\rangle \qquad [x] = \frac{q^x - q^{-x}}{q - q^{-1}} \end{aligned} \tag{22}$$

with

$$r_i \in \mathbb{Z}_+ \qquad i \in [1; n] \qquad r_i \in \{0, 1\} \qquad i \in [n+1; n+m] \qquad \sum_{i=1}^{n+m} r_i \leqslant p. \tag{23}$$

In order to write down the transformations of the basis under the action of the CAOs one has to determine the quantum analogue of the classical triple relations (15). This actually means that one has to write down the supercommutation relations between all Cartan–Weyl generators, expressed via the CAOs. The latter is a necessary condition for the application of the Poincaré–Birkhoff–Witt theorem, when computing the action of the generators on the Fock basis vectors.

Here is the result:

$$L_i\bar{L}_i = \bar{L}_iL_i = 1 \qquad L_iL_j = L_jL_i \qquad L_ia_j^\pm = q^{\mp(1+(-1)^{\theta_i}\delta_{ij})}a_j^\pm L_i \tag{24}$$

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}} \qquad [\![a_i^\eta, a_j^\eta]\!]_q = 0 \qquad \eta = \pm \qquad i < j \tag{25}$$

$$\begin{aligned} [\![[\![a_i^\eta, a_j^{-\eta}]\!], a_k^\eta]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} &= \eta^{\theta_j}\delta_{jk}L_k^{-\xi\eta}a_i^\eta + (-1)^{\theta_k}\epsilon(j,k,i)(q-\bar{q})[\![a_k^\eta, a_j^{-\eta}]\!]a_i^\eta \\ &= \eta^{\theta_j}\delta_{jk}L_k^{-\xi\eta}a_i^\eta + (-1)^{\theta_k\theta_j}\epsilon(j,k,i)q^\xi(q-\bar{q})a_i^\eta[\![a_k^\eta, a_j^{-\eta}]\!] \\ &\quad \xi(j-i) > 0 \qquad \xi,\ \eta = \pm \end{aligned} \tag{26}$$

where

$$\epsilon(j,k,i) = \begin{cases} 1 & \text{if} \quad j > k > i \\ -1 & \text{if} \quad j < k < i \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

Proposition 2. *The set of all vectors (22) constitutes an ortho-normal basis in W_p with respect to the scalar product, defined in the usual way with 'bra' and 'ket' vectors and $\langle 0|0\rangle = 1$. The transformation of the basis under the action of the CAOs reads*

$$H_i|p; r_1, r_2, \ldots, r_{n+m}) = \left(p - (-1)^{\theta_i} r_i - \sum_{j=1}^{n+m} r_j\right)|p; r_1, r_2, \ldots, r_{n+m}) \tag{28}$$

$$a_i^-|p; r_1, \ldots, r_{n+m}) = (-1)^{\theta_i(\theta_1 r_1+\cdots+\theta_{i-1}r_{i-1})} q^{r_1+\cdots+r_{i-1}} \sqrt{[r_i]\left[p - \sum_{l=1}^{n+m} r_l + 1\right]}$$
$$\times|p; r_1, \ldots r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}) \tag{29}$$

$$a_i^+|p; r_1, \ldots, r_{n+m}) = (-1)^{\theta_i(\theta_1 r_1+\cdots+\theta_{i-1}r_{i-1})} \bar{q}^{r_1+\cdots+r_{i-1}} (1 - \theta_i r_i) \sqrt{[r_i + 1]\left[p - \sum_{l=1}^{n+m} r_l\right]}$$
$$\times|p; r_1, \ldots r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_{n+m}). \tag{30}$$

The Fock representations of $sl(n+1|m)$ are obtained form the above results by replacing in equations (22), (29), (30) the quantum bracket $[\cdots]$ with a usual bracket and setting in (29) and (30) $q = 1$. The representations corresponding to an order of statistics p are irreducible and atypical representations of the Lie superalgebra $sl(n+1|m)$. In terms of Kac's classification [11], the Dynkin labels of the representation W_p are given by $(p, 0, \ldots, 0)$. This means that in general the representation W_p is multiply atypical [25]. More precisely, if $n \geqslant m$, then W_p is m-fold atypical; if $n < m$, then W_p is $(n+1)$-fold atypical for $p < m - n$ and n-fold atypical for $p \geqslant m - n$.

4. Properties of the underlying statistics

In the present paper we have recalled the definition of the superalgebras $sl(n+1|m)$ and $U_q[sl(n+1|m)]$ in terms of creation and annihilation operators. Below we justify this terminology, illustrating with a simple example how each $sl(n+1|m)$ module W_p can be viewed as a state space, where a_i^+ (respectively, a_i^-) is interpreted as an operator creating (respectively, annihilating) 'a particle' with, say, energy ε_i. For simplicity we assume that $n = m$. Let

$$b_i^{\pm} = a_i^{\pm} \qquad f_i^{\pm} = a_{i+n}^{\pm} \qquad i = 1, \ldots, n. \tag{31}$$

Consider a 'free' Hamiltonian

$$H = \sum_{i=1}^{n} \varepsilon_i (H_i + H_{i+n}) = \sum_{i=1}^{n} \varepsilon_i (\llbracket b_i^+, b_i^- \rrbracket + \llbracket f_i^+, f_i^- \rrbracket). \tag{32}$$

Then

$$[H, b_i^{\pm}] = \pm\varepsilon_i b_i^{\pm} \qquad [H, f_i^{\pm}] = \pm\varepsilon_i f_i^{\pm}. \tag{33}$$

This result together with (non-deformed) equations (29) and (30) allows one to interpret r_i, $i = 1, \ldots, n$ as the number of b-particles with energy ε_i and similarly r_{i+n}, $i = 1, \ldots, n$ as the number of f-particles with energy ε_i. Then b_i^+ (f_i^+) increases this number by one, it 'creates' a particle in the one-particle state (i.e. orbital) i. Similarly, the operator b_i^- (f_i^-) diminishes this number by one, it 'kills' a particle on the ith orbital. On every orbital i there cannot be more than one particle of kind f, whereas such a restriction does not hold for the b-particles.

These are, so to speak, Fermi-like (respectively Bose-like) properties. There is, however, one essential difference. If the order of the statistics is p then no more than p 'particles' can be accommodated in the system,

$$\sum_{i=1}^{n+m} r_i \leqslant p. \tag{34}$$

This is an immediate consequence of the transformation relation (30). Hence the filling of a given orbital depends on how many particles have already been accommodated on the other orbitals, which is neither a Bose- nor a Fermi-like property. For $\sum_{i=1}^{n+m} r_i < p$ the 'particles' behave like ordinary bosons and fermions. The maximum number of particles to be accommodated, however, in the system cannot exceed p. This is the *Pauli principle* for this statistics.

Let us consider some configurations for $m = n = 6$. Assume $p = 5$. Denote by $\bullet$ a b-particle and by $\circ$ an f-particle, and represent the six orbitals by six boxes.

(a) The state

$$|\bullet\bullet\bullet|\bullet\circ\bullet|\quad|\quad|\quad|\quad|$$

is forbidden. It is not possible to accommodate more than $p = 5$ particles.

(b) The state

$$|\bullet\circ\bullet|\circ\bullet|\quad|\quad|\quad|\quad|$$

is completely filled. It contains five particles; no more particles can be 'loaded' even in the empty 'boxes' (orbitals).

(c) The state

$$|\bullet\circ\bullet|\circ\circ|\quad|\quad|\quad|\quad|$$

is forbidden, because it contains two f-particles in the second box.

(d) Consider the state

$$|\bullet\bullet\circ|\ \bullet\ |\quad|\quad|\quad|\quad|.$$

A new b-particle can be accommodated in any box, whereas the first box is 'locked' for an f-particle. An f-particle can, however, be accommodated in any other box.

The statistics, which we have outlined above, belongs to the class of the so-called *(fractional) exclusion statistics* (ES) [26]. This issue will be considered elsewhere in more detail. The ES was introduced in an attempt to reformulate the concept of fractional statistics as *a generalized Pauli exclusion principle* for spaces with arbitrary dimension. The literature on the subject is vast, but practically in all publications one studies the thermodynamics of the ES. Here we present a microscopic description of an exclusion statistics similar to that in [27] (this is the only paper known to us attempting a microscopic description of ES). Despite the fact that exclusion statistics is defined for any space dimension, so far it was applied and tested only within one- and two-dimensional models. The statistics we have studied above are examples of a microscopic description of exclusion statistics valid in principle for any space dimension.

Acknowledgments

NIS is grateful to Professor Randjbar-Daemi for the kind invitation to visit the High Energy Section of the Abdus Salam International Centre for Theoretical Physics. TDP would like to thank the University of Ghent for a Visiting Grant, and the Department of Applied Mathematics and Computer Science for its kind hospitality during his stay in Ghent. The authors would also like to thank the referee for drawing their attention to related work.

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